choosing a frequency f_i and increasing P_i until locking occurred. Points lying on the f_i axis represent the modulation frequency obtained when the locking range is determined by the locus curve. The other points represent the modulation frequency limit when the locking range is determined by the boundary curve. For some injected frequencies ($f_i < f_0$) there are two crosses at the same frequency. The reason for this may be seen in Fig. 3(a). We look at case B with $\omega_i/\omega_0 = 0.95$ as an example. Increasing the input power from zero we find that locking occurs at the boundary curve and hence gives a cross close to $f_i - f_0$ in Fig. 4. If the input power increased further (which for a stable locked state also means increasing the voltage amplitude V) we reach the locus curve B and a jump in V (and output power) occurs. The modulation frequency just before the jump is then zero. Furthermore, this example shows the possibility of hysteresis and jumps in the output power of a locked oscillator.

V. Conclusions

Some general stability criteria have been derived that seem to be useful in connection with practical amplifiers and oscillators. The results obtained are in good agreement qualitatively with experimental results. Furthermore, a new locking figure of merit $1/Q_{ext}$ was derived for small injected powers. It is seen from this figure of merit, (27), that introducing a voltage-dependent capacitance may increase the locking bandwidth. However, this also decreases the dynamic stability of the oscillator. To increase the locking bandwidth without decreasing the dynamic stability one has to decrease the absolute value of the frequency derivative of the admittance. Furthermore, the modulation of the oscillator at the border to phase locking was studied. The difference between locking at the boundary and locus curve, respectively, was pointed out. The effect of large injected power was shown to create unsymmetric locking properties if a nonlinear susceptance is present in the circuit.

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A Wide-Band Gunn-Effect CW Waveguide Amplifier

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***Abstract*—Broad-band CW amplification with Gunn diodes in waveguide circuits has been obtained, with power gains typically between 10 and 15 dB and half-power bandwidths of more than 1 GHz. It is found that amplifier performance can be modeled with fair accuracy using a rough characterization for the diode parameters.**

Introduction

ALTHOUGH Gunn diodes have been used primarily in oscillator applications, their negative resistance properties can also be used to obtain reflection gain [1]–[8]. Three amplifying modes have been observed that depend on the product of doping density n and the length L of the GaAs chip.

McCumber and Chynoweth [9] have shown theoretically that when the nL product is less than 5×10^{11} cm^{-2} a Gunn diode exhibits a negative resistance around the transit-time frequency and its harmonics [1]. Such diodes are usually referred to as being subcritically doped, since when biased above threshold they do not enter into transit-time (Gunn) oscillations. The main disadvantage of subcritically doped amplifiers is the high ratio of capacitance-to-negative conductance that allows circuit matching for high gain only in a narrow frequency band.

When the nL product is above 5×10^{11} cm^{-2} the device is supercritically doped. If the bias voltage is just above threshold, high field domains nucleate near the cathode boundary and the diode may oscillate in the

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transit-time mode. An equivalent circuit of a Gunn diode with a high field domain present has been calculated by Hobson [10], who models the domain as a negative resistance in parallel with a capacitance. Thus a diode oscillating in the Gunn mode can be used as an amplifier by designing the circuit so as to trap the coherent oscillation. This was experimentally demonstrated by Thim [2]. The negative resistance region exists down to dc.

The third possibility is to use diodes with slightly overcritical nL products that are stable under certain conditions. Wide-band gain and substantial power outputs have been obtained in this mode [3], [4], [7], [8]. The stabilization mechanism involves device characteristics (such as field nonuniformity) and the load conditions [5]. A recent study by Magarshack and Mircea [6] suggests that the supercritically doped diodes are stabilized by electron diffusion and by small but systematic deviations from flat doping profile causing a narrow high field layer at the anode. In the experiments reported here diodes were stabilized by biasing them at 2 to 3 times the threshold voltage and by varying the load impedance at the plane of the diode. The high input bias power produces heating of the active layer and thus reduces its mobility. The net effect is a modification of the nL-product criterion of McCumber and Chynoweth [9].

All the experimental work published to date has been in coaxial and microstrip circuits where the circuit impedance levels are relatively low and can be easily transformed to the required values for diode stabilization. Another advantage of coaxial and microstrip circuits is that they are relatively dispersionless. The purpose of this paper is to report on experiments that demonstrate that supercritically doped amplifiers can be easily stabilized in waveguide circuits, although the transmission medium is dispersive and has high impedance properties.

Amplifier Circuit

The CW Gunn diode is placed into a reduced-height waveguide cavity shown in Fig. 1(a). The diode is shunt mounted in the reduced-height section, terminated by a sliding short circuit, which may be tuned to suppress oscillations and adjust the gain. A broad-band five-step transformer is used to match the reduced-height section ($b=0.0625$ in) to the full height X-band waveguide (0.400 in).

A schematic diagram of the amplifier equivalent circuit is shown in Fig. 1(b). The input impedance of the short circuit referenced to the plane of the diode is denoted by Z_s. The diode package parasitics are represented by the shunt capacitance $C_1=0.2$ pF and the lead inductance, $L_1=0.5$ nH. These values represent the average package capacitance and inductance of several

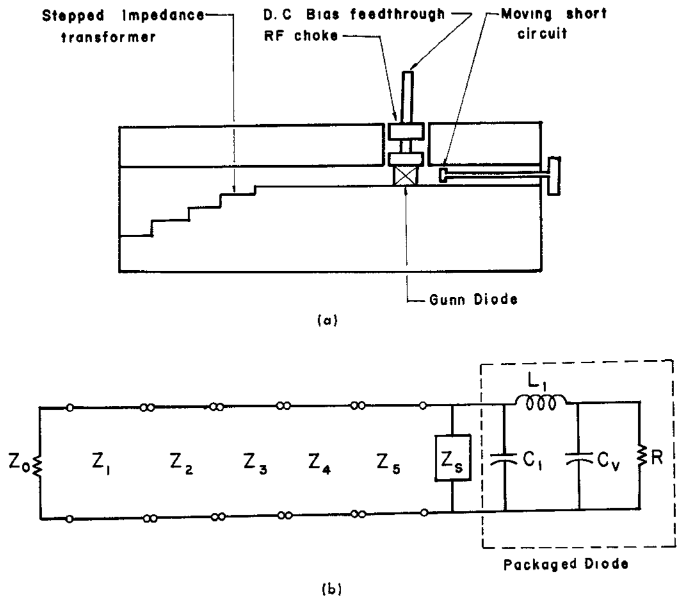

Fig. 1. (a) Cross-sectional view of reduced-height waveguide amplifier. (b) Equivalent circuit of amplifier.

similar packages measured in the same microwave circuit [12]. The transformer is modeled as a cascade of transmission lines. The load impedance is shown equal to the power-voltage characteristic impedance, Z_{01} of the standard X-band waveguide.

Frequency Response

The gain response of the Gunn amplifier is quite sensitive to bias voltage. Fig. 2(a) shows the gain at three different voltages for a fixed position of the short circuit. At frequencies below 11 GHz the gain increases with voltage. Above 11 GHz it drops slightly. At 10 GHz the voltage sensitivity of the gain is 8 dB/V, at 9 GHz it is 7 dB/V. At 8 V the gain exceeds 11 dB over a 1.4-GHz bandwidth; the corresponding 3-dB bandwidth is about 2 GHz.

The phase shift between the incident and reflected signals was measured referenced to the input flange of the amplifier circuit. The phase-shift dependence on bias voltage is shown in Fig. 2(b). Below 9 GHz the phase response is linear within experimental error. The phase shift increases by about 40°/V. At midband there is some nonlinearity. The changes in phase response are due to the dependence of the diode impedance on voltage.

Fig. 3(a) shows the variations of the gain with temperature at 7.5 V and the short circuit held at 0.2415 in. At the low end of the band the gain increases with temperature. At 9 GHz there is a 4-dB increase in the gain as the temperature is varied from 80°F to 122°F. Note the striking similarity between the temperature and the voltage dependance of the gain. The sensitivity of the phase response to temperature variations

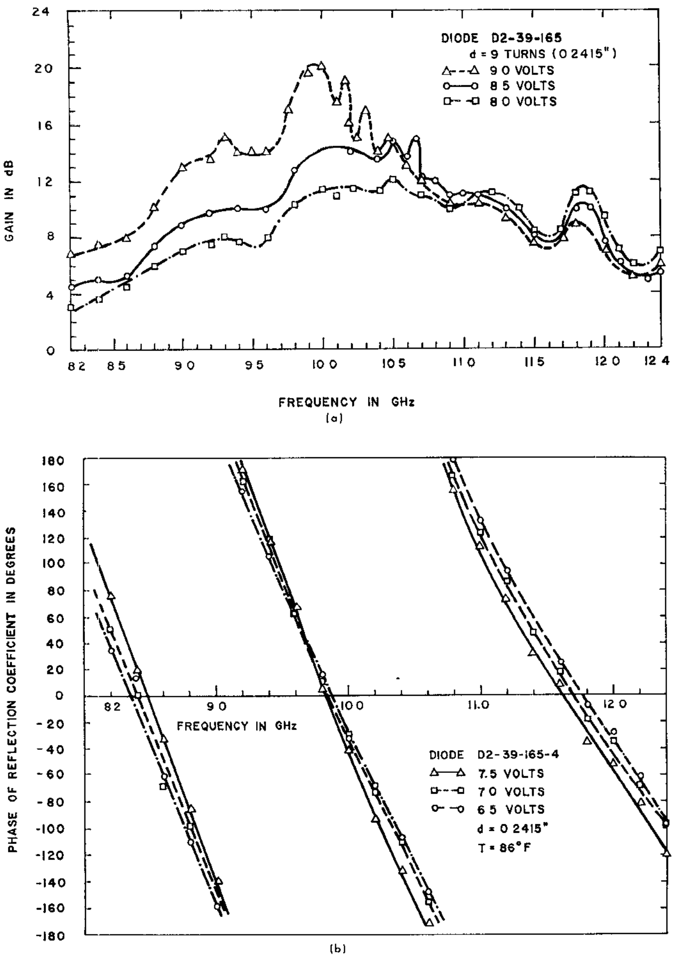

Fig. 2. Frequency response with bias voltage as parameter. (a) Gain. (b) Phase.

is shown in Fig. 3(b). In this experiment the phase shift decreases by about 1.4°/°C.

The heak-sink temperature was increased to 150°F, but the diode oscillated and could not be stabilized by changing either the voltage or the tuning short position.

Diode Characterization

Using the equivalent circuit of Fig. 1(b) and experimental data of the phase and gain response of the amplifier, the equivalent circuit parameters of the diode can be calculated. Let Z_0 be the characteristic impedance of the output waveguide and Z the input impedance of the amplifier, then the reflection coefficient at the input of the amplifier is:

$$\Gamma = \frac{Z - Z_0}{Z + Z_0}. \tag{1}$$

Knowing $|\Gamma|$ and arg $(\Gamma) = \phi$, the input impedance of the amplifier can be shown to be

$$Z(\omega) = R(\omega) + jX(\omega) \tag{2}$$

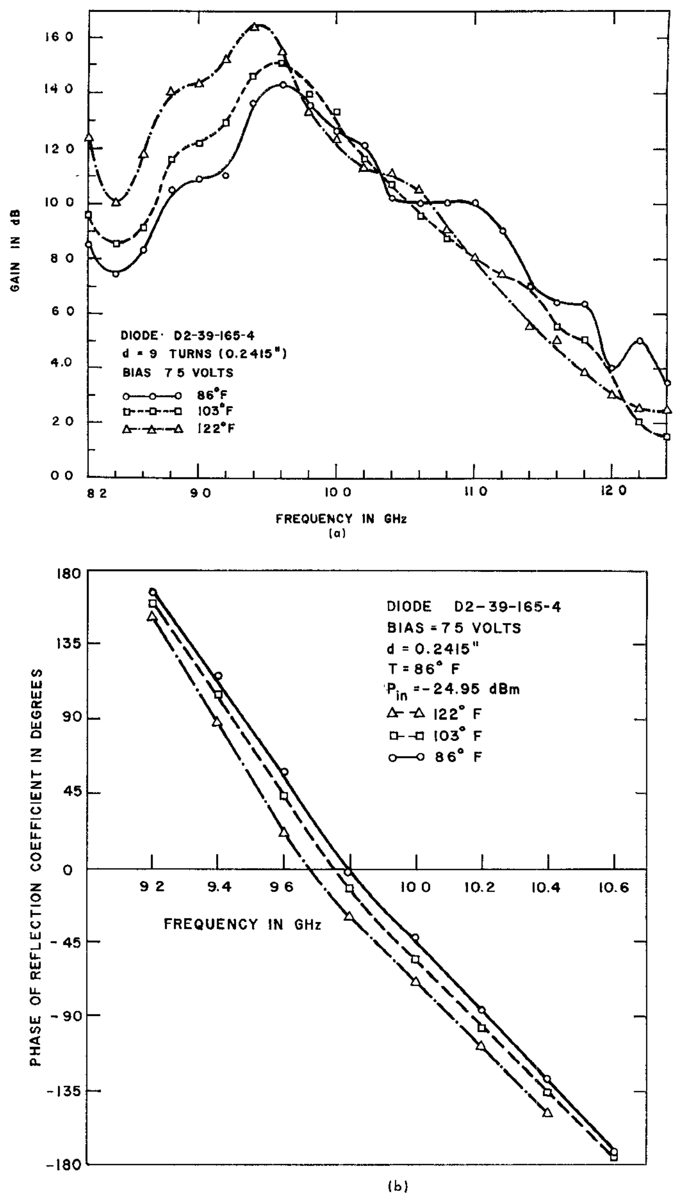

Fig. 3. Frequency response with heat-sink temperature as parameter. (a) Gain. (b) Phase.

where

$$R(\omega) = Z_0 \frac{1 - |\Gamma|^2}{1 + |\Gamma|^2 - 2|\Gamma| \cos \phi} \tag{3}$$

$$X(\omega) = 2Z_0 \frac{|\Gamma| \sin \phi}{1 + |\Gamma|^2 - 2|\Gamma| \cos \phi}. \tag{4}$$

Since in a reflection amplifier $|\Gamma| > 1$, it is seen that the input impedance has a negative resistance component.

The sensitivity of the diode series resistance to voltage is shown in Fig. 4. The negative resistance at 6.5 V is relatively flat at $-30\ \Omega$ over a 1-GHz bandwidth, while at 7.0 V it is nearly constant at $-42\ \Omega$ over 1.2 GHz. The increase in the magnitude of the series resistance corresponds to an increasing gain with bias voltage.

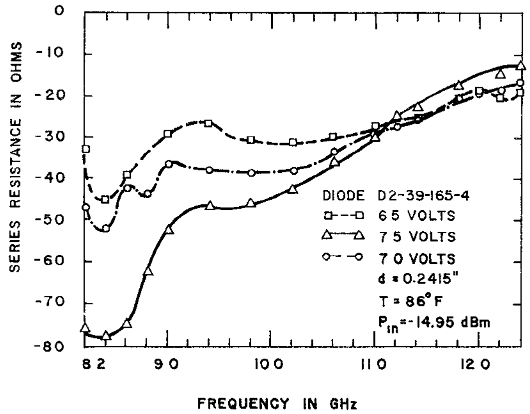

Fig. 4. Experimental series equivalent resistance for Gunn diode with bias voltage as parameter.

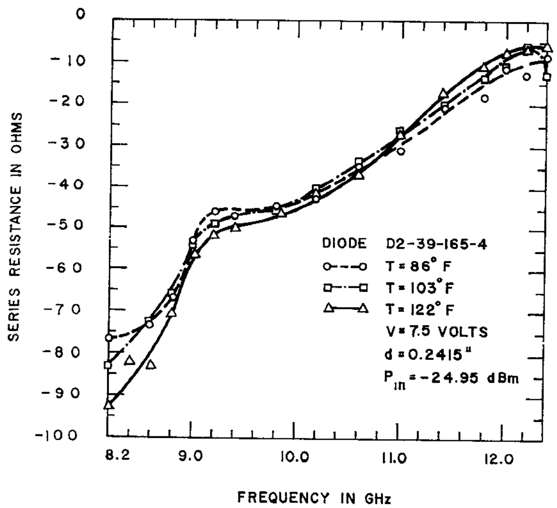

Fig. 5. Experimental series resistance of Gunn diode with heat-sink temperature as parameter.

However, as the voltage increases, the series negative resistance may become equal to the series load, in which case the circuit becomes unstable if there is a resonance between the circuit and diode reactances. This was experimentally seen on some diodes when the bias approached 10 V. The dependence of the series negative resistance of the diode on temperature is shown in Fig. 5. Again it is seen that voltage and temperature have a similar effect on the diode behavior.

Attempts to calculate the diode series capacitance gave unreliable results. Notice that errors in the phase measurement can affect the apparent series reactance drastically due to the sine term in (4).

The Gunn-diode amplifier can be modeled as a series negative resistance and capacitance, both of which are frequency dependent. Fig. 4 shows that at 6.5-V bias the series negative resistance is flat at about $-30\,\Omega$ over a 1-GHz bandwidth. Using this value and a series capacitance of 0.5 pF (i.e., the cold-diode parallel-plate capacitance) a theoretical frequency response was calculated. Fig. 6 shows a comparison of the theoretical

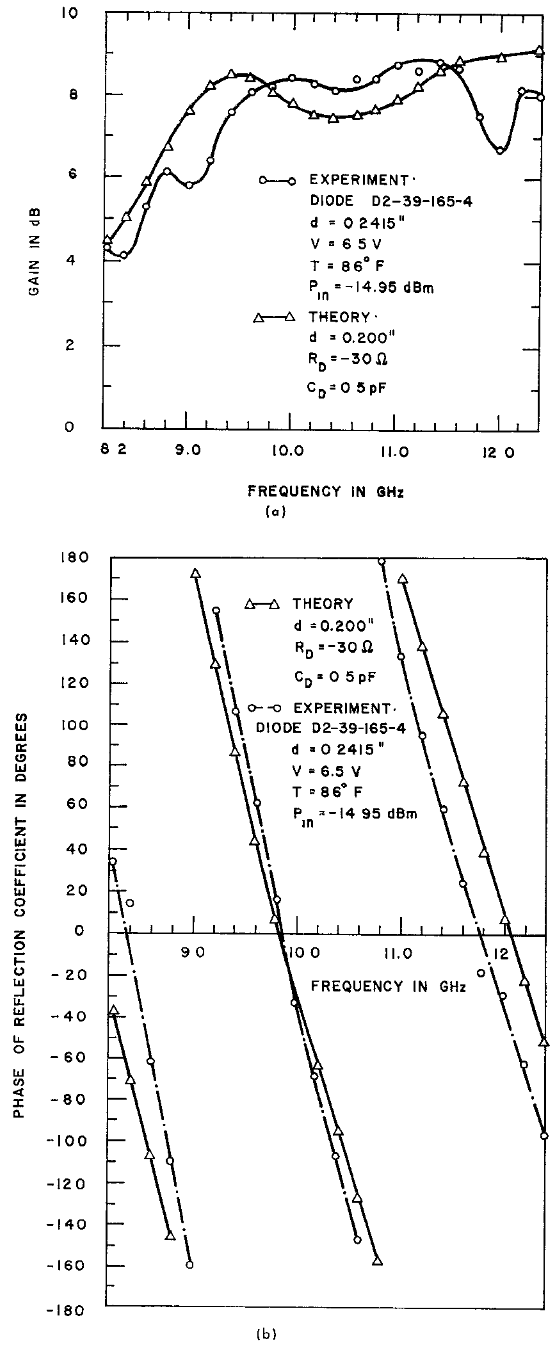

Fig. 6. Comparison of theoretical and experimental frequency response. (a) Gain. (b) Phase.

and experimental results. The theoretical tuning short position was adjusted slightly to account for circuit modeling errors and to improve the fit.

The good agreement noted here suggests that even with a rough characterization of the diode, it is possible to model the amplifier response and to optimize a Gunn-amplifier design in the same way transistor amplifiers are now optimized. For example, the effect of varying the short-circuit plane from 0.1 to 0.5 in from the plane of the diode is shown in Fig. 7(a) and (b). As the amplifier is tuned, the frequency for maximum

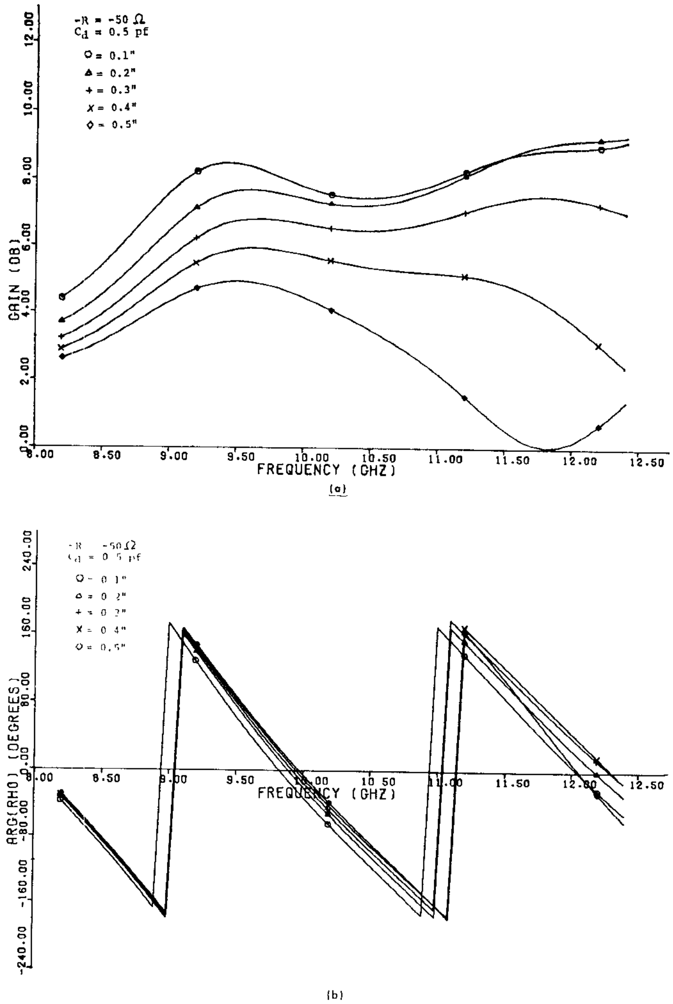

Fig. 7. Theoretical frequency response with tuning short position as parameter. (a) Gain. (b) Phase.

gain moves toward the low end of the band. Such tuning of the frequency response has been observed experimentally.

Performance

The variation of the linear range with bias voltage is shown in Fig. 8. The frequency is 10 GHz. At 8.5 V the small signal gain is 15.6 dB; 1-dB gain compression occurs at −5-dBm input power. At 9 V the gain is 18.6 dB, but the linear range is considerably reduced. The gain becomes nonlinear at −15-dBm input power. A maximum linear dynamic range of 94 dB has been measured.

The effect on the phase response of varying the input power was measured and is shown in Fig. 9. As gain compression is approached the phase shift of the reflected signal increases when compared to the small signal value. A figure of merit of about 2°/dB just below the 1-dB-gain compression point was observed.

The noise figure of the amplifier was measured. When the diode was stabilized and biased for broad-band gain response, it was found that it generated 3 μW of

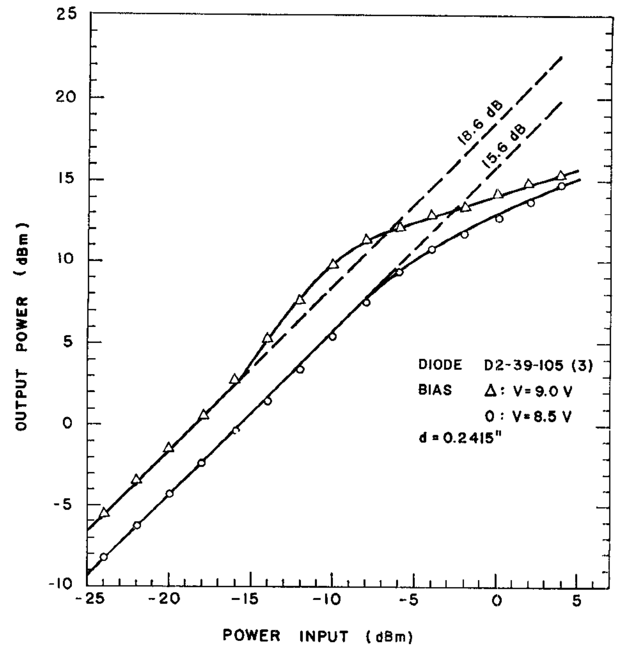

Fig. 8. Dynamic range and saturation behavior of amplifier.

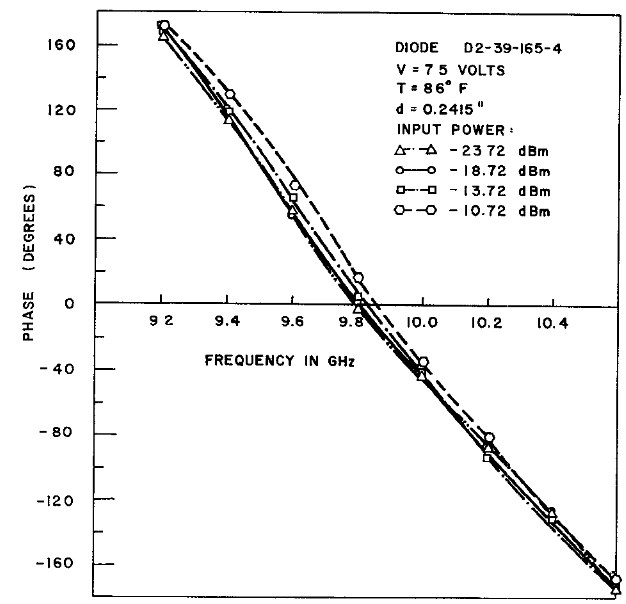

Fig. 9. Phase response with input signal power as parameter.

noise power. No coherent oscillations could be detected with a spectrum analyzer. This broad-band noise is likely due to the incoherent formation of space charge in the active layer of the Gunn diode [13], [14]. Assuming that the noise generated by the diode is white, its power spectral density is 3.2×10^{-16} W/Hz over the waveguide bandwidth. In order to make the noise figure measurement, the operating point of the diode was optimized by varying the bias voltage and the short-circuit position until no noise power can be detected. This tuning resulted in a narrow-band gain response centered at 8.74 GHz. The gain was 10 dB and the amplifier noise figure was 25 dB. Noise figures in the range 15 to 20 dB have been reported [7] for coaxial amplifiers.

The amplifier results reported here were obtained by using a 10-dB directional coupler to monitor the output. The load VSWR at the amplifier input was less than 1.2:1 over the X band.

Discussion

The similarity between the effects of bias voltage and heat-sink temperature indicate a reduction in carrier mobility due to thermal effects as part of the stabilization mechanism of supercritically doped Gunn-diode amplifiers. Such diodes proved to be easily stabilized in waveguide circuits that could provide low enough values of load resistance. For the circuit shown in Fig. 1 the calculated load resistance decreased from 140 Ω at 8 GHz to 42 Ω at 12.4 GHz.

The demonstration of stable operation of Gunn-diode amplifiers in waveguide circuits opens the possibility of solid-state amplifiers at millimeter wavelengths.

Acknowledgment

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Finite-Element Solutions within Curved Boundaries

DAVID J. RICHARDS AND ALVIN WEXLER

Abstract—The paper shows that a curved boundary need not be approximated by a small number of finite-element sides, resulting in a coarse polygonal approximation to the shape of the region and consequent inaccuracies, but may be defined as accurately as desired. An algorithm and associated mathematics are presented for locating the stationary point of a functional by the Rayleigh–Ritz method with a two-variable power series as a trial function. As a particular example, the functional employed is one that is made stationary by the solution of Poisson's equation under mixed, Dirichlet, or Neumann boundary conditions. The technique is based on the fact that the three boundary conditions are natural ones. Results are presented for a problem involving curved boundaries under mixed and Neumann conditions and for the capacitance calculations of a pair of noncoaxial cylinders having specified potentials. Comparisons are made with the finite-difference method. It is concluded that the finite-element method is, in nearly all aspects, superior to finite differences—particularly when curved boundary modeling errors are reduced. It is expected that the method described will be equally useful for, and quite simple to adapt to, the solution of the Helmholtz equation in an enclosed region.

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